3.3 Discover the Exponent Laws

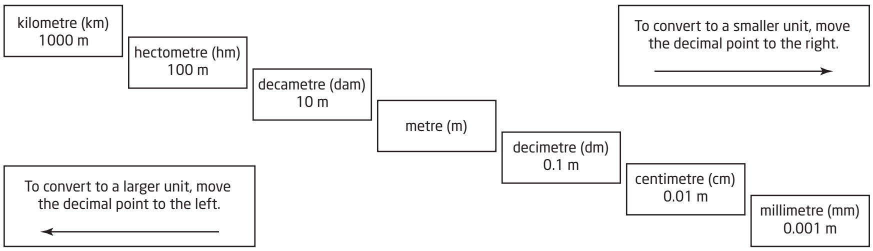
The 100-m dash is one of the most exciting events in track and field. If you ran this race, how many centimetres would you run? How many millimetres is this?

Investigate A

How can you simplify expressions involving products and quotients of powers?

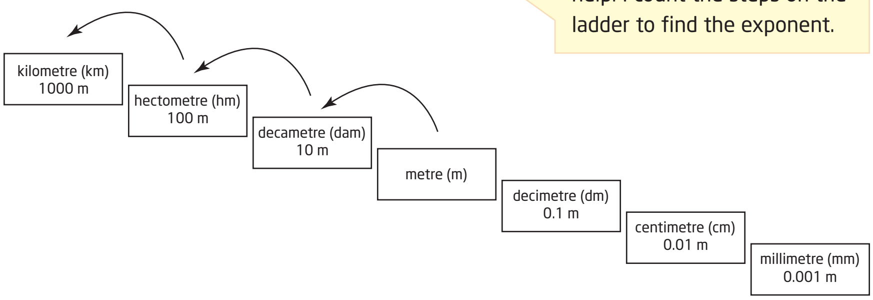
Part A: Patterns Involving Powers of 10

In the metric system, length measures are related by powers of 10. For example, there are 10 mm in 1 cm. This makes it easy to convert one unit of length to another. Note that hectometres and decametres are uncommon units.



1. How many metres are in 1 km? Write this as a power of 10.

I can use the metric ladder to help. I count the steps on the ladder to find the exponent.



2. Copy and complete the table.

Unit	Number of these in 1 m	Power of 10
decimetre	10	10^1
centimetre	100	
millimetre		

Did You Know?

A Canadian named Donovan Bailey set the men's world record for the 100-m dash at the 1996 Olympics. He ran the race in 9.84 s, and was considered at that time to be the world's fastest human.

3. a) Multiply the number of centimetres in 1 m by the number of metres in 1 km. What does this answer give you?
   b) Write the product in part a) using powers of 10. Write the answer as a power of 10.

4. Repeat step 3 for millimetres instead of centimetres.

5. **Reflect** Look at the exponents in the powers of 10 in your answers to steps 3b) and 4b). Describe how these numbers are related.

Part B: Products of Powers

How can you simplify expressions containing products of powers with the same base?

6. Copy and complete the table, including an example of your own.

Quotient	Expanded Form	Single Power
$3^2 \times 3^4$	$(3 \times 3) \times (3 \times 3 \times 3 \times 3)$	3^6
$4^3 \times 4^3$		
$6^4 \times 6^1$		
$2^4 \times 2^2 \times 2^3$		
$k^3 \times k^5$		
Create your own example		

7. What do you notice about the bases of the powers in each product in the first column?

8. Look at the exponents in the first column for each product. How does the sum of the exponents compare to the exponent in the last column?

9. **Reflect** Explain how you can write a product of powers using a single power. Use your example to illustrate your explanation.

10. Write a rule for finding the product of powers by copying and completing the equation $x^a \times x^b = \blacksquare$.

Part C: Quotients of Powers

How can you simplify expressions containing quotients of powers with the same base?

11. Copy and complete the table, including an example of your own.

I can reduce common factors.

Quotient	Expanded Form	Single Power
$5^5 \div 5^3$	$\frac{5 \times 5 \times \cancel{5} \times \cancel{5} \times \cancel{5}}{\cancel{5} \times \cancel{5} \times \cancel{5}}$	5^2
$7^4 \div 7^1$		
$10^6 \div 10^4$		
$2^7 \div 2^6$		
$p^8 \div p^5$		
Create your own example		

12. What do you notice about the bases of the powers in each quotient in the first column?

13. Look at the exponents in the first column for each quotient. How do they relate to the exponent of the single power in the last column?

14. **Reflect** Explain how you can write a quotient of powers using a single power. Use your example to illustrate your explanation.

15. Write a rule for finding the quotient of powers by copying and completing the equation $x^a \div x^b = \blacksquare$.

The patterns in the activity above illustrate two **exponent laws**. The exponent laws are a set of rules that allow you to simplify expressions involving powers with *the same base*.

Product Rule

When multiplying powers with *the same base*, add the exponents to write the product as a single power:
$x^a \times x^b = x^{a+b}$

Quotient Rule

When dividing powers with *the same base*, subtract the exponents to write the quotient as a single power:
$x^a \div x^b = x^{a-b}$

Example 1 Apply the Product Rule

Write each product as a single power. Then, evaluate the power.

a) $3^2 \times 3^3$

b) $5^2 \times 5 \times 5^2$

c) $(-2)^4 \times (-2)^3$

d) $\left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^2$

e) $0.1^4 \times 0.1^2$

Solution

a) $3^2 \times 3^3$
$= 3^{2+3}$
$= 3^5$
$= 243$

The bases are the same, so I can add the exponents.

C 3 y^x 5 =

b) $5^2 \times 5 \times 5^2$
$= 5^2 \times 5^1 \times 5^2$
$= 5^5$
$= 3125$

When no exponent appears, I know that it is 1.

$5^2 \times 5 \times 5^2 = 5^2 \times 5^1 \times 5^2$

Now, I can add the exponents: $2 + 1 + 2 = 5$

c) $(-2)^4 \times (-2)^3$
$= (-2)^{4+3}$
$= (-2)^7$
$= -128$

d) $\left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^2$

$$= \left(\frac{1}{2}\right)^{3+2}$$

$$= \left(\frac{1}{2}\right)^5$$

$$= \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right)$$

$$= \frac{1 \times 1 \times 1 \times 1 \times 1}{2 \times 2 \times 2 \times 2 \times 2}$$

$$= \frac{1}{32}$$

e) $0.1^4 \times 0.1^2$
$= 0.1^{4+2}$
$= 0.1^6$
$= 0.000\ 001$

To find the sixth power of 0.1, I need to multiply 0.1 by itself six times. In the product, there will be six digits after the decimal point.

Example 2 Apply the Quotient Rule

Write each product as a single power. Then, evaluate the power.

a) $8^7 \div 8^5$

b) $4^7 \div 4 \div 4^3$

c) $\dfrac{(-0.5)^6}{(-0.5)^3}$

d) $\dfrac{\left(\frac{3}{4}\right)^3 \times \left(\frac{3}{4}\right)^2}{\left(\frac{3}{4}\right)^5}$

Solution

a) $8^7 \div 8^5$
$= 8^{7-5}$
$= 8^2$
$= 64$

The bases are the same, so I can subtract the exponents.

b) $4^7 \div 4 \div 4^3$
$= 4^7 \div 4^1 \div 4^3$
$= 4^{7-1} \div 4^3$
$= 4^6 \div 4^3$
$= 4^{6-3}$
$= 4^3$
$= 64$

Divide in order from left to right.

c) $\dfrac{(-0.5)^6}{(-0.5)^3}$
$= (-0.5)^{6-3}$
$= (-0.5)^3$
$= -0.125$

d) $\dfrac{\left(\frac{3}{4}\right)^3 \times \left(\frac{3}{4}\right)^2}{\left(\frac{3}{4}\right)^5}$

$= \dfrac{\left(\frac{3}{4}\right)^{3+2}}{\left(\frac{3}{4}\right)^5}$

Apply the product rule first to simplify the numerator.

$= \dfrac{\left(\frac{3}{4}\right)^5}{\left(\frac{3}{4}\right)^5}$

Anything divided by itself equals 1.

$= 1$

What if I use the quotient rule?

$\dfrac{\left(\frac{3}{4}\right)^5}{\left(\frac{3}{4}\right)^5} = \left(\frac{3}{4}\right)^{5-5}$

$= \left(\frac{3}{4}\right)^0$

I know the answer is 1.
I wonder if an exponent of 0 always gives an answer of 1?

Investigate B

How can you simplify expressions involving powers of powers?

1. Copy and complete the table, including an example of your own.

Power of a Power	Expanded Form	Single Power
$(2^2)^3$	$(2^2) \times (2^2) \times (2^2)$ $= (2 \times 2) \times (2 \times 2) \times (2 \times 2)$	2^6
$(5^3)^4$		
$(10^4)^2$		
Create your own example		

2. Look at the exponents in the first column for each case. How do they relate to the exponent of the single power in the last column?

3. **Reflect** Explain how you can write a power of a power using a single power. Use your example to illustrate your explanation.

4. Write a rule for finding the power of a power by copying and completing the equation $(x^a)^b = \blacksquare$.

The patterns in Investigate B illustrate another exponent law.

Power of a Power Rule
A power of a power can be written as a single power by multiplying the exponents.
$(x^a)^b = x^{a \times b}$

Example 3 Apply the Power of a Power Rule

Write each as a single power. Then, evaluate the power.

a) $(3^2)^4$

b) $[(-2)^3]^4$

c) $\left[\left(\frac{2}{3}\right)^2\right]^2$

d) $(0.2^3)^2$

Solution

a)
$$\begin{aligned}(3^2)^4 &= 3^{2 \times 4} \\ &= 3^8 \\ &= 6561\end{aligned}$$

b)
$$\begin{aligned}[(-2)^3]^4 &= (-2)^{3 \times 4} \\ &= (-2)^{12} \\ &= 4096\end{aligned}$$

c)
$$\begin{aligned}\left[\left(\frac{2}{3}\right)^2\right]^2 &= \left(\frac{2}{3}\right)^{2 \times 2} \\ &= \left(\frac{2}{3}\right)^4 \\ &= \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \\ &= \frac{16}{81}\end{aligned}$$

d)
$$\begin{aligned}(0.2^3)^2 &= 0.2^{3 \times 2} \\ &= 0.2^6 \\ &= 0.000\,064\end{aligned}$$

Example 4 Simplify Algebraic Expressions

Simplify each algebraic expression by applying the exponent laws.

a) $y^3 \times y^5$

b) $6p^7 \div 3p^3$

c) $a^2b^3 \times a^6b^4$

d) $\dfrac{-2uv^3 \times 8u^2v^2}{(4uv^2)^2}$

Solution

a) $y^3 \times y^5 = y^{3+5}$ **Apply the product rule.**

$= y^8$

b) $6p^7 \div 3p^3 = 2p^{7-3}$ **Divide the numeric factors, 6 ÷ 3 = 2.**

$= 2p^4$

c) $a^2b^3 \times a^6b^1 = a^8b^4$

Exponent laws only apply to powers with the same base.
- First, I add exponents of a: $a^{2+6} = a^8$
- Then, I add exponents of b: $b^{3+1} = b^4$

d) $\dfrac{-2uv^3 \times 8u^2v^2}{(4uv^2)^2}$ **Simplify numerator and denominator first. The exponent in the denominator applies to all the factors inside the brackets.**

$$= \frac{(-2) \times 8 \times u \times u^2 \times v^3 \times v^2}{4^2u^2(v^2)^2}$$

$$= \frac{-16 \times u^{1+2} \times v^{3+2}}{16u^2v^{2\times 2}}$$

$$= \frac{-16u^3v^5}{16u^2v^4}$$ **Divide.**

$$= -u^{3-2}v^{5-4}$$

$$= -uv$$

Key Concepts

- The exponent laws are a way to simplify expressions involving powers with the same base.
- When multiplying powers with the same base, add the exponents: $x^a \times x^b = x^{a+b}$
- When dividing powers with the same base, subtract the exponents: $x^a \div x^b = x^{a-b}$
- When finding the power of a power, multiply the exponents: $(x^a)^b = x^{a \times b}$

Communicate Your Understanding

C1 Identify which exponent law you can apply to simplify each expression. If no exponent law can be used, explain why not.

a) $6^3 \div 6^2$

b) $(m^2)^3$

c) $3^4 \times 4^3$

d) $a^2b \times a^3b^4$

e) $\dfrac{p^3q^2}{pq}$

f) $\dfrac{u^4v^5}{w^2x^3}$

C2 Create an example involving powers where you can

a) add exponents

b) multiply exponents

c) subtract exponents

C3 Look at part d) of Example 4. Suppose that $u = 3$ and $v = -2$:

Original expression	Simplified expression
$\dfrac{-2uv^3 \times 8u^2v^2}{(4uv^2)^2}$	$-uv$

a) Which expression would you rather substitute into to evaluate the expression, and why?

b) What is the value of the expression after substituting the given values?

Practise

For help with questions 1 and 2, see Example 1.

1. Which is $7^3 \times 7^2$ expressed as a single power?

A 7^6 **B** 7^5

C 7^9 **D** 49^6

2. Apply the product rule to write each as a single power. Then, evaluate the expression.

a) $3^4 \times 3^7$ **b)** $2^4 \times 2 \times 2^3$

c) $(-1)^5 \times (-1)^6$ **d)** $\left(\dfrac{2}{5}\right)^3 \times \left(\dfrac{2}{5}\right)^3$

For help with questions 3 and 4, see Example 2.

3. Which is $11^7 \div 11^5$ expressed as a single power?

A 11^{12} **B** $11^{1.4}$ **C** 1^2 **D** 11^2

4. Apply the quotient rule to write each as a single power. Then, evaluate the expression.

a) $12^8 \div 12^2$ **b)** $(-6)^5 \div (-6)^2 \div (-6)^2$

c) $\left(-\frac{3}{4}\right)^4 \div \left(-\frac{3}{4}\right)$ **d)** $\frac{0.1^6 \div 0.1^4}{0.1^2}$

For help with questions 5 and 6, see Example 3.

5. Which is $(5^4)^2$ expressed as a single power?

A 5^8 **B** 5^6

C 25^6 **D** 25^8

6. Apply the power of a power rule to write each as a single power. Then, evaluate the expression.

a) $(4^2)^2$ **b)** $[(-3)^3]^2$

c) $[(-0.1)^4]^2$ **d)** $\left[\left(\frac{3}{2}\right)^3\right]^2$

7. Simplify using the exponent laws. Then, evaluate.

a) $5^2 \times 5^3 \div 5^4$ **b)** $3^7 \div 3^5 \times 3$

c) $\frac{(0.5^3)^4}{0.5^6 \times 0.5^4}$ **d)** $(-2)^4 \times (-2)^5 \div [(-2)^3]^3$

For help with questions 8 and 9, see Example 4.

8. Simplify.

a) $y^4 \times y^2$ **b)** $m^8 \div m^5$ **c)** $k^2 \times k^3 \times k^5$

d) $(c^3)^4$ **e)** $a^2b^2 \times a^3b$ **f)** $(2uv^2)^3$

g) $m^2n \times mn^2$ **h)** $h^2k^3 \div hk$ **i)** $(-a^3b)^2$

Connect and Apply

9. Simplify.

a) $12k^2m^8 \div 4km^5$ **b)** $-8a^5 \times (2a^3)^2$ **c)** $(-x^2)^3 \times (3x^2)^2$

d) $\frac{4d^4w^3 \times 6dw^4}{3d^3w \times 8dw^2}$ **e)** $\frac{3f^4g^3 \times 8fg^4}{(6f^2g^3)^2}$ **f)** $(3a^2b)^2 \div (ab)^2$

g) $\frac{5c^3d \times 4c^2d^2}{(2c^2d)^2}$ **h)** $\frac{(3xy^2)^3 \times (-4x^2y)}{(2x^2y^2)^2}$ **i)** $\frac{30g^2h \times (2gh)^2}{5gh^2 \times 6gh}$

Did You Know?

The notation that we use for powers, with a raised number for the exponent, was invented by Réné Descartes (1596-1650). Descartes used this notation in his text Géométrie, published in 1637. In this famous text, Descartes connected algebra and geometry, starting the branch of mathematics called Cartesian geometry.

10. Consider the expression $\dfrac{5xy^2 \times 2x^2y}{(2xy)^2}$.

a) Substitute $x = 3$ and $y = -1$ into the expression. Then, evaluate the expression.

b) Simplify the original expression using the exponent laws. Then, substitute the given values and evaluate the expression.

c) Describe the advantages and disadvantages of each method.

Literacy Connections

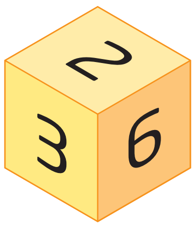
Rectangular prism is the mathematical name for a box.

11. A crawlspace in a space station has the shape of a rectangular prism. It is about 100 cm high, 10 m wide, and 1 km long. What is the volume enclosed by the crawlspace?

12. The probability of tossing heads with a standard coin is $\frac{1}{2}$, because it is one of two possible outcomes. The probability of tossing three heads in a row is $\left(\frac{1}{2}\right)^3$ or $\frac{1}{8}$.

a) What is the probability of tossing
- six heads in a row?
- 12 heads in a row?

b) Write each answer in part a) as a power of a power.



13. a) What is the probability of rolling a 6 with a standard number cube?

b) What is the probability of rolling four 6s in a row?

c) What is the probability of rolling a perfect square with a number cube?

d) What is the probability of rolling eight perfect squares in a row?

e) Write each answer in parts b) and d) as a power of a power.

Achievement Check

14. Consider the expression $\dfrac{-3m^2n \times 4m^3n^2}{(2m^2n)^2 \times 3mn}$.

a) Substitute $m = 4$ and $n = -3$ into the expression and evaluate it.

b) Simplify the original expression using the exponent laws.

c) Substitute $m = 4$ and $n = -3$ into the simplified expression and evaluate it.

d) What did you notice? What are the advantages and disadvantages of the two methods?

e) Josie made two errors in copying the above expression. She wrote $\dfrac{-3m^2n \times 4mn^2}{(2mn)^2 \times 3mn}$, but she still got the correct answer. Explain how this is possible.

Extend

15. You can multiply and divide numbers in scientific notation by applying the exponent laws. For example,

$$
\begin{aligned}
&(3 \times 10^5) \times (2 \times 10^4)\\
&= 3 \times 2 \times 10^5 \times 10^4\\
&= 6 \times 10^9
\end{aligned}
$$

$$
\begin{aligned}
&(9 \times 10^8) \div (3 \times 10^5)\\
&= \frac{9 \times 10^8}{3 \times 10^5}\\
&= \frac{\overset{3}{\cancel{9}} \times 10^8}{\underset{1}{\cancel{3}} \times 10^5}\\
&= 3 \times 10^3
\end{aligned}
$$

Evaluate each of the following. Express each answer in scientific notation and then in standard notation.

a) $4 \times 10^2 \times 2 \times 10^3$

b) $1.5 \times 10^4 \times 6 \times 10^6$

c) $(8 \times 10^7) \div (2 \times 10^5)$

d) $(3.9 \times 10^{12}) \div (3 \times 10^8)$

16. a) Predict the screen output of your scientific or graphing calculator when you enter the following calculation: $(3 \times 10^5) \div (2 \times 10^6)$.

b) Is the answer what you predicted? Explain the answer that the calculator has provided.

17. a) Predict the screen output of your scientific or graphing calculator when you enter the following calculation: $(3 \times 10^{18}) \div (6 \times 10^2)$.

b) Is the answer what you predicted? Explain the answer that the calculator has provided.

18. a) Evaluate $(2 \times 10^5)^3$. Express your answer in both scientific and standard notation.

b) Explain how you can evaluate a power of a number expressed in scientific notation. Create an example of your own to help illustrate your explanation.

19. Math Contest Copy and complete the table to make the square a multiplicative magic square (the product of every row, column, and diagonal is equal).

$a^{19}b^{16}$		$a^{15}b^{12}$
a^9b^6	$a^{13}b^{10}$	
	$a^{21}b^{18}$	a^7b^4

20. Math Contest If $x = \frac{1}{9}$, place the following values in order from least to greatest:

$x, x^2, x^3, \sqrt{x}, \frac{1}{x}$